

THE COLORED SYMMETRIC AND EXTERIOR ALGEBRAS

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ABSTRACT. We study colored generalizations of the symmetric algebra and its Koszul dual, the exterior algebra. The symmetric group \mathfrak{S}_n acts on the multilinear components of these algebras. While \mathfrak{S}_n acts trivially on the multilinear components of the colored symmetric algebra, we use poset topology techniques to understand the representation on its Koszul dual. We introduce an \mathfrak{S}_n -poset of weighted subsets that we call the weighted boolean algebra and we prove that the multilinear components of the colored exterior algebra are \mathfrak{S}_n -isomorphic to the top cohomology modules of its maximal intervals. We use a technique of Sundaram to compute group representations on Cohen-macaulay posets to give a generating formula for the Frobenius series of the colored exterior algebra. We exploit that formula to find an explicit expression for the expansion of the corresponding representations in terms of irreducible \mathfrak{S}_n -representations. We show that the two colored Koszul dual algebras are Koszul in the sense of Priddy et al.

1. INTRODUCTION

Let \mathbf{k} denote an arbitrary field of characteristic not equal 2 and V be a finite dimensional \mathbf{k} -vector space. The *tensor algebra* $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$ is the free associative algebra generated by V , where $V^{\otimes n}$ denotes the tensor product of n copies of V and where $V^{\otimes 0} := \mathbf{k}$. For any set $R \subseteq T(V)$ denote by $\langle R \rangle$ the ideal of $T(V)$ generated by R . Let R_1 be the subspace of $V \otimes V$ generated by the set of relations of the form

$$(1.1) \quad x \otimes y - y \otimes x \quad (\text{symmetry}),$$

for all $x, y \in V$. The *symmetric algebra* $\mathcal{S}(V)$ is the quotient algebra

$$\mathcal{S}(V) := T(V) / \langle R_1 \rangle.$$

Now let $R_2 \subseteq V \otimes V$ be generated by the set of relations of the form

$$(1.2) \quad x \otimes y + y \otimes x \quad (\text{antisymmetry}),$$

for all $x, y \in V$. The *exterior algebra* $\Lambda(V)$ is the algebra

$$\Lambda(V) = T(V) / \langle R_2 \rangle.$$

We will use the concatenation xy to denote the image of $x \otimes y$ in $\mathcal{S}(V)$ and the *wedge* $x \wedge y$ to denote the image of $x \otimes y$ in $\Lambda(V)$ under the canonical epimorphisms.

Let $V^* := \text{Hom}(V, \mathbf{k})$ denote the vector space dual to V . For finite dimensional V we have that $V^* \simeq V$. Recall that for an associative algebra $A = A(V, R) := T(V)/\langle R \rangle$ generated on a finite dimensional vector space V and (quadratic) relations $R \subseteq V^{\otimes 2}$ there is another algebra A^\dagger associated to A that is called the *Koszul dual associative algebra* to A . Indeed, when V is finite dimensional, there is a canonical isomorphism $(V^{\otimes 2})^* \simeq V^* \otimes V^*$ and we let R^\perp be the image under this isomorphism of the space of elements in $(V^{\otimes 2})^*$ that vanish on R . The *Koszul dual* A^\dagger of A is the algebra $A^\dagger := A(V^*, R^\perp) = T(V^*)/\langle R^\perp \rangle$. It is known and easy to check from the relations (1.1) and (1.2) (see for example [13]) that $\Lambda(V^*)$ is the Koszul dual associative algebra to $\mathcal{S}(V)$.

Denote $[n] := \{1, 2, \dots, n\}$ and let $V = \mathbf{k}[n]$ be the vector space with generators $[n]$. We define the *multilinear component* $\mathcal{S}(n)$ as the subspace of $\mathcal{S}(V)$ generated by products of the form $\sigma(1)\sigma(2)\cdots\sigma(n)$ where σ is a permutation in the symmetric group \mathfrak{S}_n . Similarly $\Lambda(n)$ is defined to be the subspace of $\Lambda(V)$ generated by *wedged permutations*, i.e., the generators are of the form $\sigma(1) \wedge \sigma(2) \wedge \cdots \wedge \sigma(n)$ for $\sigma \in \mathfrak{S}_n$. The symmetric group acts on the generators of $\mathcal{S}(n)$ and $\Lambda(n)$ by permuting their letters and this action induces representations of \mathfrak{S}_n in both $\mathcal{S}(n)$ and $\Lambda(n)$. Using the relations (1.1) and (1.2) we can see that both $\mathcal{S}(n)$ and $\Lambda(n)$ are always one-dimensional spaces with bases given by $\{12\cdots n\}$ and $\{1 \wedge 2 \wedge \cdots \wedge n\}$ respectively. Moreover, for $n \geq 1$ it is easy to see that

$$\mathcal{S}(n) \cong_{\mathfrak{S}_n} \mathbf{1}_n \text{ and } \Lambda(n) \cong_{\mathfrak{S}_n} \text{sgn}_n,$$

where $\mathbf{1}_n$ and sgn_n are respectively the trivial and the sign representations of \mathfrak{S}_n .

1.1. Colored symmetric and exterior algebras. Let \mathbb{N} denote the set of nonnegative integers and \mathbb{P} the set of positive integers. For a subset $S \subseteq \mathbb{P}$ we consider the set $[n]^S := [n] \times S$ of colored letters of the form (x, i) (that we will denote x^i) where $x \in [n]$ and $i \in S$. Let $V = \mathbf{k}[n]$ and $V^S = \mathbf{k}[n]^S$ where $S \subseteq \mathbb{P}$ is finite and $CR_1 \subseteq V^S \otimes V^S$ be generated by

$$(1.3) \quad x^i \otimes y^j - y^i \otimes x^j \quad (\text{label symmetry}),$$

$$(1.4) \quad x^i \otimes y^j - x^j \otimes y^i \quad (\text{color symmetry}),$$

for all $x, y \in [n]$ and $i, j \in S$. The S -colored symmetric algebra $\mathcal{S}_S(V)$ is defined to be the algebra

$$\mathcal{S}_S(V) := T(V)/\langle CR_1 \rangle.$$

We define the S -colored exterior algebra $\Lambda_S(V^*)$ on V^* as the Koszul dual to $\mathcal{S}_S(V)$. Explicitly, the reader can check that if we let $CR_2 \subseteq V^S \otimes V^S$ be generated by

$$(1.5) \quad x^i \otimes y^i + y^i \otimes x^i \quad (\text{monochromatic antisymmetry}),$$

$$(1.6) \quad x^i \otimes y^j + y^i \otimes x^j + y^j \otimes x^i + x^j \otimes y^i \quad (\text{mixed antisymmetry}),$$

for all $x, y \in [n]$ and $i, j \in S$, then

$$\Lambda_S(V) := T(V)/\langle CR_2 \rangle.$$

Choosing $S = [k]$ for some $k \in \mathbb{P}$ and letting k be large we obtain the *colored symmetric algebra* $\mathcal{S}_{\mathbb{P}}(V)$ and the *colored exterior algebra* $\Lambda_{\mathbb{P}}(V)$. We denote $\mathcal{S}_{\mathbb{P}}(n)$ and $\Lambda_{\mathbb{P}}(n)$ respectively the components of $\mathcal{S}_{\mathbb{P}}(V)$ and $\Lambda_{\mathbb{P}}(V)$ generated by colored permutations and wedged colored permutations. A *colored permutation* is a permutation $\sigma \in \mathfrak{S}_n$ together with a function that assigns to each $x \in [n]$ a *color* $\mathbf{color}(x) \in \mathbb{P}$. For example $2^1 1^4 3^2$ is a colored permutation of $[3]$ (here $\mathbf{color}(1) = 4$, $\mathbf{color}(2) = 1$ and $\mathbf{color}(3) = 2$) and so a generator in $\mathcal{S}_{\mathbb{P}}(3)$. For a colored permutation σ of n let $\wedge(\sigma)$ denote the *wedged colored permutation* $\sigma(1) \wedge \sigma(2) \wedge \cdots \wedge \sigma(n)$. For example $\wedge(2^1 1^4 3^2) = 2^1 \wedge 1^4 \wedge 3^2$ is a generator in $\Lambda_{\mathbb{P}}(3)$. Let \mathfrak{S}_n^S denote the set of colored permutations of $[n]$ with colors in $S \subseteq \mathbb{P}$.

A *weak composition* μ of n is a sequence of nonnegative integers $(\mu(1), \mu(2), \dots)$ such that $|\mu| := \sum_{i \geq 1} \mu(i) = n$. Let wcomp be the set of weak compositions and wcomp_n the set of weak compositions of n . For example, $(0, 1, 2, 0, 1) := (0, 1, 2, 0, 1, 0, 0, \dots)$ is in wcomp_4 . In this work weak compositions will be the combinatorial device used to track the number of occurrences of each color among the letters of the generators. Indeed, for $\sigma \in \mathfrak{S}_n^{\mathbb{P}}$, let $\mu(\sigma) \in \text{wcomp}$ be such that $\mu(\sigma)(j)$ is the number of letters of color j in σ for each j , we call $\mu(\sigma)$ the *content* of σ . For example $\mu(2^3 1^5 3^2 4^3) = (0, 1, 2, 0, 1)$. For $\mu \in \text{wcomp}$ we denote by \mathfrak{S}_{μ} the set of colored permutations of content μ . Define $\mathcal{S}(\mu)$ and $\Lambda(\mu)$ to be respectively the subspaces of $\mathcal{S}_{\mathbb{P}}(|\mu|)$ and $\Lambda_{\mathbb{P}}(|\mu|)$ generated by colored permutations and wedged colored permutations in \mathfrak{S}_{μ} . For example $\mathcal{S}(0, 1, 2, 0, 1)$ and $\Lambda(0, 1, 2, 0, 1)$ have generators associated with colored permutations of $[4]$ that contain one letter of color 2, two letters of color 3, one letter of color 5 and no other letters. The symmetric group \mathfrak{S}_n acts on $\mathcal{S}_{\mathbb{P}}(n)$ and $\Lambda_{\mathbb{P}}(n)$ as before.

A permutation $\tau \in \mathfrak{S}_n$ acts by replacing the colored letter x^i for the colored letter $\tau(x)^i$. This action preserves the colors of the generators and so $\mathcal{S}(\mu)$ and $\Lambda(\mu)$ are also representations of \mathfrak{S}_n . Naturally if ν is a rearrangement of μ , we have that $\mathcal{S}(\nu) \simeq_{\mathfrak{S}_n} \mathcal{S}(\mu)$ and $\Lambda(\nu) \simeq_{\mathfrak{S}_n} \Lambda(\mu)$. In particular, if μ has a single nonzero component then $\mathcal{S}(\mu)$ is isomorphic to $\mathcal{S}(n)$ and $\Lambda(\mu)$ is isomorphic to $\Lambda(n)$.

For $\mu \in \text{wcomp}_n$ define its *support* $\text{supp}(\mu) = \{j \in \mathbb{P} \mid \mu(j) \neq 0\}$. Then for $S \subseteq \mathbb{P}$ we have

$$\mathcal{S}_S(n) \simeq_{\mathfrak{S}_n} \bigoplus_{\substack{\mu \in \text{wcomp}_n \\ \text{supp}(\mu) \subseteq S}} \mathcal{S}(\mu) \quad \text{and} \quad \Lambda_S(n) \simeq_{\mathfrak{S}_n} \bigoplus_{\substack{\mu \in \text{wcomp}_n \\ \text{supp}(\mu) \subseteq S}} \Lambda(\mu).$$

The following theorem follows immediately from relations (1.3) and (1.4).

Theorem 1.1. *For $n \geq 1$ and $\mu \in \text{wcomp}_n$,*

$$\mathcal{S}(\mu) \cong_{\mathfrak{S}_n} \mathbf{1}_n.$$

Our goal is to understand the more interesting representation of \mathfrak{S}_n on $\Lambda(\mu)$ for all $\mu \in \text{wcomp}$. In order to accomplish this we are going to apply the program started by Hanlon and Wachs in [19] and by Wachs in [36]. They applied poset topology techniques to the partially ordered set (or *poset*) Π_n of partitions of the set $[n]$, and to the induced subposet of Π_n where all partitions have parts of size congruent to 1 mod $(k-1)$, in order to understand algebraic properties of the multilinear components of the free Lie algebra and the free Lie k -algebra (see also [2]). Gottlieb and Wachs [18] have extended the results on the poset of partitions to more general Dowling lattices. The author and Wachs [17] and the author [16] have applied similar techniques to a family of posets of weighted partitions in their study of the operad of Lie algebras with multiple compatible brackets and its Koszul dual operad, the operad of commutative algebras with multiple totally commutative products (see also [8, 22]). The original motivation for the present work is precisely the study of analogous constructions within the category of connected graded associative algebras.

The main idea of the technique in [19, 36, 18, 17, 16] is that in order to study the representation of \mathfrak{S}_n on the multilinear component $\Lambda(n)$ of certain algebra A , a certain poset P_A is constructed so that the (co)homology of P_A (defined later) is \mathfrak{S}_n -isomorphic to $\Lambda(n)$ (maybe up to tensoring with the sign representation). Then poset topology techniques applied to P_A can recover algebraic information about $\Lambda(n)$.

2. MAIN RESULTS

To every poset P one can associate a simplicial complex $\Delta(P)$ (called the *order complex*) whose faces are the chains (totally ordered subsets) of P . If there is a group G acting on the poset in such a way that every $g \in G$ is a (strict) order preserving map on P then this action induces isomorphic representations of G on the j -th reduced simplicial homology $\tilde{H}_j(P)$ and cohomology $\tilde{H}^j(P)$ of the order complex $\Delta(P)$. Let P be *bounded* (it has a minimal element (denoted $\hat{0}$) and a maximal element (denoted $\hat{1}$)) and *pure* (all the maximal chains have the same length), if for every open interval (x, y) in P it happens that $\tilde{H}_i((x, y)) = 0$ for all $x < y$ in P and $i < l([x, y]) - 2$ we say that P is *Cohen-Macaulay*. Some poset topology techniques on pure bounded posets, like the theory of lexicographic shellability, imply Cohen-Macaulayness.

Recall that the *boolean algebra* \mathbb{B}_n is the poset of subsets of $[n]$ ordered by inclusion. It is known that \mathbb{B}_n is Cohen-Macaulay (see for example [3]) and there is a natural action of \mathfrak{S}_n on \mathbb{B}_n permuting the elements in $[n]$. This action induces a representation of \mathfrak{S}_n on the unique nonvanishing reduced simplicial cohomology $\tilde{H}^{n-2}(\overline{\mathbb{B}}_n)$ of the proper part $\overline{\mathbb{B}}_n := \mathbb{B}_n \setminus \{\hat{0}, \hat{1}\}$ of \mathbb{B}_n . The following isomorphism is already a classical result (see [31]),

$$(2.1) \quad \tilde{H}^{n-2}(\overline{\mathbb{B}}_n) \cong_{\mathfrak{S}_n} \text{sgn}_n \quad (\cong_{\mathfrak{S}_n} \Lambda(n)).$$

To use the technique mentioned above we need to construct the poset P_A associated to the algebra A such that the (co)homology of P_A is \mathfrak{S}_n -isomorphic to $\Lambda(n)$. For certain algebras there is a recipe to cook up the poset P_A . Méndez and Yang [25] have developed a way to associate a family of posets to an injective monoid in the monoidal categories of species with respect to the operations of product and composition. Vallette [35] have rediscovered the construction in [25] for basic set operads (injective monoids with respect to composition) and used it to develop a criterion for the Koszulness of the associated operad and its Koszul dual operad by checking that the maximal intervals in the associated posets are Cohen-Macaulay (see also Fresse [11]). An operad is an algebraic structure that encodes types of algebras (see [23]). In analogy with [35], Méndez in [26] have developed a criterion for Koszulness of an associative algebra (with the left cancellative property) and its Koszul dual associative algebra.

We will be following very closely the thread of ideas in [16]. We will recall here some of the concepts involved while referring the reader to consult [16] for most of the background and related notation.

2.1. The weighted boolean algebra. For undefined poset notation and terminology the reader can consult [33]. Let \mathbf{WCOMP} be the partially ordered set of weak compositions with order relation defined as follows: for every $\nu, \mu \in \mathbf{wcomp}$, we say that $\mu \leq \nu$ if $\mu(i) \leq \nu(i)$ for every i . We define \mathbf{WCOMP}_n to be the induced subposet of \mathbf{WCOMP} whose elements are weak compositions $\mu \in \mathbf{wcomp}$ with $|\mu| \leq n$. A *weighted subset* of $[n]$ is a set B^μ where $B \subseteq [n]$ and $\mu \in \mathbf{wcomp}_{|B|}$. Since weak compositions are infinite vectors we can use component-wise addition and subtraction, for instance, we denote by $\nu + \mu$, the weak composition defined by $(\nu + \mu)(i) := \nu(i) + \mu(i)$.

The *weighted boolean algebra* \mathbb{B}_n^w is the partially ordered set (poset) of weighted subsets of $[n]$ with order relation given by $A^\mu \leq B^\nu$ if the following conditions hold:

- $A \leq B$ in \mathbb{B}_n and,
- $\mu \leq \nu$ in \mathbf{WCOMP}_n .

Equivalently, we can define the covering relation $A^\mu \lessdot B^\nu$ by:

- $A \lessdot B$ in \mathbb{B}_n and,
- $\nu - \mu = \mathbf{e}_r$ for some $r \in \mathbb{P}$, where \mathbf{e}_r is the weak composition with a 1 in the r -th component and 0 in all other entries.

For $S \in \mathbb{P}$ we also denote by \mathbb{B}_n^S the induced subposet of \mathbb{B}_n^w whose elements are weighted subsets B^μ with $\text{supp}(\mu) \subseteq S$.

In Figure 1 we illustrate the *Hasse diagram* of the poset $\mathbb{B}_3^{[2]}$.

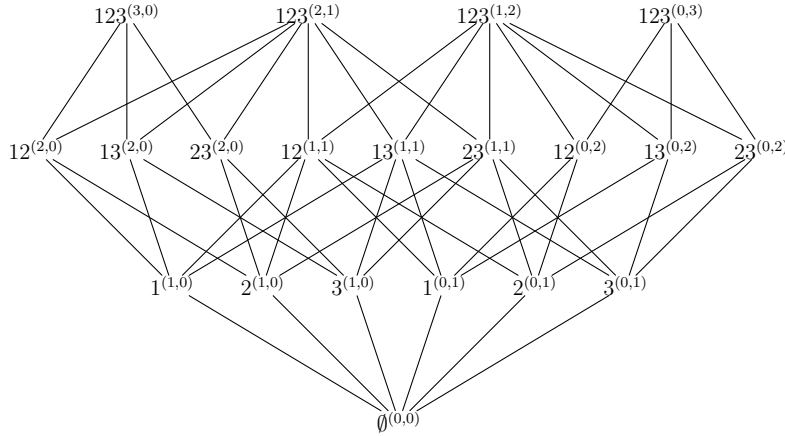


FIGURE 1. $\mathbb{B}_3^{[2]}$

We can define \mathbb{B}_n^w in a different way. Note that both posets \mathbb{B}_n and \mathbf{WCOMP}_n are ranked and hence have well-defined poset maps $rk : \mathbb{B}_n \rightarrow C_{n+1}$ and $rk : \mathbf{WCOMP}_n \rightarrow C_{n+1}$ to the chain with $n + 1$

elements C_{n+1} . Recall that the *Segre or fiber product* $P_{f,g}^\times Q$ of two poset maps $f : P \rightarrow R$ and $g : Q \rightarrow R$ is the induced subposet of the product $P \times Q$ with elements $\{(p, q) \mid f(p) = g(q)\}$. Then we have that $\mathbb{B}_n^w = \mathbb{B}_{nrk, rk}^\times \text{WCOMP}_n$.

The poset \mathbb{B}_n^w has a minimum element $\hat{0} := \emptyset^0$ and maximal elements $[n]^\mu := \{[n]^\mu\}$ indexed by weak compositions $\mu \in \text{wcomp}_n$. Note that for every $\nu, \mu \in \text{wcomp}_n$ such that ν is a rearrangement of μ , the maximal intervals $[\hat{0}, [n]^\nu]$ and $[\hat{0}, [n]^\mu]$ are isomorphic to each other. In particular, if μ has a single nonzero component, these intervals are isomorphic to \mathbb{B}_n , hence $\mathbb{B}_n^{[1]} \simeq \mathbb{B}_n$. In the case when $S = [2]$ the poset $\mathbb{B}_n^{[2]}$ is isomorphic to a poset introduced by Shareshian and Wachs in [30] and it is closely related to a poset of Björner and Welker in [6].

The symmetric group \mathfrak{S}_n acts on \mathbb{B}_n^w in the following way: for any $B^\mu \in \mathbb{B}_n^w$ and $\tau \in \mathfrak{S}_n$ we have $\tau B^\mu := (\tau B)^\mu$ where $\tau B := \{\tau(i) \mid i \in B\}$. Since any $\tau \in \mathfrak{S}_n$ is a strict order preserving morphism, the action of \mathfrak{S}_n on \mathbb{B}_n^w induces an action on the unique nonzero reduced (co)homology $\tilde{H}^{n-2}((\hat{0}, [n]^\mu))$ of the open maximal interval $(\hat{0}, [n]^\mu)$ of \mathbb{B}_n^w . In Section 3 we prove the following isomorphism (Theorem 3.4).

Theorem 2.1. *For $n \geq 0$ and $\mu \in \text{wcomp}_n$,*

$$\Lambda(\mu) \simeq_{\mathfrak{S}_n} \tilde{H}^{n-2}((\hat{0}, [n]^\mu)).$$

An *EL-labeling* is a certain type of labeling of the edges of the *Hasse diagram* of a poset P that satisfy certain requirements (defined in Section 4.1). EL-labelings were introduced by Björner [3] and further developed by Björner and Wachs [4]. A poset P that admits an EL-labeling is said to be *EL-shellable* and EL-shellability has important topological implications on its order complex $\Delta(P)$. Let $\widehat{\mathbb{B}}_n^w := \mathbb{B}_n^w \cup \{\hat{1}\}$ be the poset \mathbb{B}_n^w after a maximal element has been added. In Section 4 we prove the following theorem.

Theorem 2.2. *$\widehat{\mathbb{B}}_n^w$ is EL-shellable and hence Cohen-Macaulay.*

An *ascent* in a colored permutation $\sigma \in \mathfrak{S}_n^{\mathbb{P}}$ is a value $i \in [n-1]$ such that

- $\tilde{\sigma}(i) < \tilde{\sigma}(i+1)$, that is, i is an ascent in the underlying uncolored permutation $\tilde{\sigma} \in \mathfrak{S}_n$, and
- $\text{color}(\sigma(i)) \leq \text{color}(\sigma(i+1))$.

A *nonincreasing colored permutation* is a colored permutation $\sigma \in \mathfrak{S}_n^{\mathbb{P}}$ that is ascent free.

For example $2^1 1^4 3^2$ is a nonincreasing colored permutation but $2^1 1^2 3^4$ is not since the pair $(1^2, 3^4)$ forms an ascent. Let Ninc_n be the set of

nonincreasing colored permutations and \mathbf{Ninc}_μ the ones with content μ . In Section 4 using the EL-labeling mentioned in Theorem 2.2 and results in the theory of lexicographic shellability ([3, 4]) we prove that:

Theorem 2.3. *For $\mu \in \text{wcomp}$ the set*

$$\{\wedge(\sigma) \mid \sigma \in \mathbf{Ninc}_\mu\}$$

is a basis for $\Lambda(\mu)$. Consequently,

$$\dim \Lambda(\mu) = |\mathbf{Ninc}_\mu|.$$

Consider now the generating function

$$(2.2) \quad \sum_{\mu \in \text{wcomp}_n} \dim \Lambda(\mu) \mathbf{x}^\mu,$$

where $\mathbf{x}^\mu = x_1^{\mu(1)} x_2^{\mu(2)} \cdots$. Since $\dim \Lambda(\mu)$ is invariant under any rearrangement of the parts of μ we have that (2.2) belongs to the ring of symmetric functions $\Lambda_{\mathbb{Z}}$ (see [24] and [32, Chapter 7] for the definitions).

The symmetric function (2.2) is also e -nonnegative; i.e., the coefficients of its expansion in the basis of elementary symmetric functions are all nonnegative. Indeed, we associate a *type* (or integer partition) to each $\sigma \in \mathfrak{S}_n$ in the following way: Let $\pi(\sigma)$ be the finest (set) partition of the set $[n]$ satisfying

- whenever $\sigma(i) < \sigma(i+1)$ for some $i \in [n-1]$, $\sigma(i)$ and $\sigma(i+1)$ belong to the same block of $\pi(\sigma)$.

We define the *type* $\lambda(\sigma)$ of σ to be the (integer) partition whose parts are the sizes of the blocks of $\pi(\sigma)$. For example, for the permutation 5126473 the associated partition is $\lambda(\sigma) = (3, 2, 1, 1)$.

Theorem 2.4. *For all n ,*

$$\sum_{\mu \in \text{wcomp}_n} \dim \Lambda(\mu) \mathbf{x}^\mu = \sum_{\sigma \in \mathfrak{S}_n} e_{\lambda(\sigma)}(\mathbf{x}),$$

where e_λ is the elementary symmetric function associated with the partition λ .

The following theorem gives another characterization of this symmetric function.

Theorem 2.5. *We have*

$$\sum_{n \geq 0} \sum_{\mu \in \text{wcomp}_n} \dim \Lambda(\mu) \mathbf{x}^\mu \frac{y^n}{n!} = \left[\sum_{n \geq 0} (-1)^n h_n(\mathbf{x}) \frac{y^n}{n!} \right]^{-1},$$

where h_n is the complete homogeneous symmetric function and $(\cdot)^{-1}$ denotes the multiplicative inverse of a formal power series.

Even though Theorem 2.5 can be proven directly (using for example the recursive definition of the Möbius invariant of the maximal intervals of \mathbb{B}_n^w , Philip Hall's theorem and the isomorphism of Theorem 2.1), in Section 5 we prove, using a technique of Sundaram [34] to compute group representations on Cohen-Macaulay G -posets, an equivariant version that reduces to Theorem 2.5 by specialization. Let $\text{ch } V$ denote the Frobenius characteristic in variables $\mathbf{y} = (y_1, y_2, \dots)$ of an \mathfrak{S}_n -module V . Recall that the map ch is an isomorphism between the Grothendieck ring of representations of symmetric groups $\text{Rep}_{\mathfrak{S}}$ and the ring of symmetric functions $\Lambda_{\mathbb{Z}}$ where the Schur function s_{λ} is the image under ch of the Specht module (irreducible \mathfrak{S}_n -module) S^{λ} .

Theorem 2.6. *We have that*

$$\sum_{n \geq 0} \sum_{\mu \in \text{wcomp}_n} \text{ch } \Lambda(\mu) \mathbf{x}^{\mu} = \left(\sum_{n \geq 0} (-1)^n h_n(\mathbf{x}) h_n(\mathbf{y}) \right)^{-1}.$$

Note that the power series in Theorem 2.6 belongs to the ring of symmetric power series in \mathbf{y} with coefficients in the ring $\Lambda_{\mathbb{Q}}$ of symmetric functions in \mathbf{x} .

We use a theorem discovered by Fröberg [12]; Carlitz, Scoville and Vaughan [7]; and Gessel [14] that provides an explicit description of the multiplicative inverse in Theorem 2.6 to give the following explicit formula for the representation of \mathfrak{S}_n on $\Lambda(\mu)$.

Theorem 2.7. *We have that*

$$\sum_{\mu \in \text{wcomp}_n} \text{ch } \Lambda(\mu) \mathbf{x}^{\mu} = \sum_{\eta \vdash n} \sum_{\alpha \in \text{comp}_n} c_{H(\alpha), \eta} e_{\lambda(\alpha)}(\mathbf{x}) s_{\eta}(\mathbf{y})$$

where $c_{H(\alpha), \eta}$ is the number of Young tableaux of shape η and descent set $\text{des } H(\alpha)$ (defined in Section 5) and comp_n is the set of integer compositions of n .

Theorem 2.7 can be seen as an equivariant version of the nonnegativity in the e basis in Theorem 2.4. Indeed, if we write

$$\sum_{\mu \in \text{wcomp}_n} \text{ch } \Lambda(\mu) \mathbf{x}^{\mu} = \sum_{\lambda \vdash n} C_{\lambda}(\mathbf{y}) e_{\lambda}(\mathbf{x}),$$

then Theorem 2.7 implies that the coefficients $C_{\lambda}(\mathbf{y})$ are Schur-nonnegative.

In Section 6 we describe (Theorem 6.2) how the technique developed by Méndez in [26] implies that for a finite S and finite dimensional V the Koszul dual algebras $\mathcal{S}_S(V)$ and $\Lambda_S(V)$ are Koszul in the sense of Priddy [27].

Finally, in Section 7 we show how certain specializations of the identity in Theorem 2.6, and the identity obtained from this together with Theorem 2.7, reduce to a classical formula for the exponential generating function of the Eulerian polynomials and also to an identity that involves the characters of the regular representations of \mathfrak{S}_n for all n . The latter one revealing that the representation $\Lambda_S(n)$ is a generalization of the regular representation $\mathbb{C}[\mathfrak{S}_n]$.

3. THE ISOMORPHISM

3.1. A generating set for $\tilde{H}^{n-2}(\hat{0}, [n]^\mu)$. The top dimensional cohomology of a pure poset P , say of length ℓ , has a particularly simple description. Let $\mathcal{M}(P)$ denote the set of maximal chains of P and let $\mathcal{M}'(P)$ denote the set of chains of length $\ell-1$. We view the coboundary map δ as a map from the chain space of P to itself, which takes chains of length r to chains of length $r+1$ for all r . It can be shown (see for example the appendix in [17]) that δ acts on r -chains as follows:

$$\delta_r(\alpha_0 < \cdots < \alpha_r) = \sum_{i=0}^{r+1} (-1)^i \sum_{\alpha \in (\alpha_{i-1}, \alpha_i)} (\alpha_0 < \cdots < \alpha_{i-1} < \alpha < \alpha_i < \cdots < \alpha_r),$$

where $\alpha_{-1} = \hat{0}$ and $\alpha_{r+1} = \hat{1}$. Since the image of δ on the top chain space (i.e. the space spanned by $\mathcal{M}(P)$) is 0, the kernel is the entire top chain space. Hence top cohomology is the quotient of the space spanned by $\mathcal{M}(P)$ by the image of the space spanned by $\mathcal{M}'(P)$. The image of $\mathcal{M}'(P)$ is what we call the coboundary relations. We thus have the following presentation of the top cohomology

$$\tilde{H}^\ell(P) = \langle \mathcal{M}(P) \mid \text{coboundary relations} \rangle.$$

Note that for any $\alpha = \alpha_0 < \cdots < \alpha_{\ell-1} \in \mathcal{M}'(P)$ there is exactly one step $\alpha_{i-1} < \alpha_i$ for some $i = 0, \dots, \ell$ where the chain α can be *refined* (or augmented) to get a chain in $\mathcal{M}(P)$. Then the cohomology relations in the top cohomology are generated by relations of the form

$$(3.1) \quad \sum_{\alpha \in (\alpha_{i-1}, \alpha_i)} (\alpha_0 < \cdots < \alpha_{i-1} < \alpha < \alpha_i < \cdots < \alpha_d) = 0.$$

3.1.1. Description of the maximal chains. We show that the maximal chains in a maximal interval $[\hat{0}, [n]^\mu]$ of \mathbb{B}_n^w are in bijection with colored permutations, hence we can use permutations in \mathfrak{S}_μ to describe the elements of $\mathcal{M}([\hat{0}, [n]^\mu])$. A map $\bar{\lambda} : \mathcal{E}(P) \rightarrow \Lambda$, where $\mathcal{E}(P)$ is the set of edges (covering relations) of the Hasse diagram of a poset P and Λ is

a fixed poset is called an *edge labeling*. Note that a covering relation in \mathbb{B}_n^w is of the form $A^\nu \lessdot (A \cup \{x\})^{\nu + \mathbf{e}_i}$ where A^ν is a weighted subset of $[n]$, $x \in [n] \setminus A$ and $i \in \mathbb{P}$. So we can associate a labeling $\bar{\lambda} : \mathcal{E}(\mathbb{B}_n^w) \rightarrow [n]^\mathbb{P}$ given by

$$(3.2) \quad \bar{\lambda}(A^\nu, (A \cup \{x\})^{\nu + \mathbf{e}_i}) = x^i,$$

where $[n]^\mathbb{P} := [n] \times \mathbb{P}$ is the product poset of the totally ordered sets $[n]$ and \mathbb{P} . In Section 4 we conclude that this labeling $\bar{\lambda}$ is actually an EL-labeling of \mathbb{B}_n^w . Furthermore, this labeling can be extended to an EL-labeling of $\widehat{\mathbb{B}}_n^w := \mathbb{B}_n^w \cup \{\hat{1}\}$ (\mathbb{B}_n^w with a maximal element added). We denote by

$$\bar{\lambda}(c) = \bar{\lambda}(x_0, x_1) \bar{\lambda}(x_1, x_2) \cdots \bar{\lambda}(x_{\ell-1}, x_\ell),$$

the word of labels corresponding to a maximal chain $c = (\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{\ell-1} \lessdot x_\ell = \hat{1})$. In the case $c \in \mathcal{M}([\hat{0}, [n]^\mu])$ it is immediate that $\bar{\lambda}(c) \in \mathfrak{S}_\mu$ since in c each letter of $[n]$ appears exactly once and the possible sequences of colors in c are determined by μ . For example the chain

$$\hat{0} \lessdot \{2\}^{(1,0,0,0)} \lessdot \{1, 2\}^{(1,0,0,1)} \lessdot \{1, 2, 3\}^{(1,1,0,1)}$$

corresponds to the word of labels $2^1 1^4 3^2$. Clearly, starting with $\sigma \in \mathfrak{S}_\mu$ we can also recover the chain $c \in \mathcal{M}([\hat{0}, [n]^\mu])$ such that $\bar{\lambda}(c) = \sigma$. Indeed, for $\sigma \in \mathfrak{S}_\mu$ define the chain $c(\sigma) \in \mathcal{M}([\hat{0}, [n]^\mu])$ to be the one whose rank 0 element is $\hat{0}$ and whose rank i weighted subset is

$$\{\sigma(1), \sigma(2), \dots, \sigma(i)\}^{\mathbf{e}_{\text{color}(\sigma(1))} + \mathbf{e}_{\text{color}(\sigma(2))} + \cdots + \mathbf{e}_{\text{color}(\sigma(i))}}$$

for all $i \in [n]$. We conclude the following theorem.

Proposition 3.1. *The maps $\bar{\lambda}$ and c above define a bijection*

$$\mathcal{M}([\hat{0}, [n]^\mu]) \simeq \mathfrak{S}_\mu.$$

Note also that for a bounded poset P the sets $\mathcal{M}(P)$ and $\mathcal{M}(P \setminus \{\hat{0}, \hat{1}\})$ are in bijection by associating a chain $c \in \mathcal{M}(P)$ with the chain $\bar{c} := c \setminus \{\hat{0}, \hat{1}\} \in \mathcal{M}(P \setminus \{\hat{0}, \hat{1}\})$. For $\sigma \in \mathfrak{S}_\mu$ we write $\bar{c}(\sigma) := c(\sigma) \setminus \{\hat{0}, \hat{1}\}$ for the corresponding chain in $(\hat{0}, [n]^\mu)$.

The codimension one chains in $(\hat{0}, [n]^\mu)$ are unrefinable except between a pair of adjacent elements in $[\hat{0}, [n]^\mu]$ so the generating relations of equation (3.1) correspond to the two different type of intervals of length 2 in $[\hat{0}, [n]^\mu]$. The intervals of length 2 happen when two elements x and y have been added to a weighted subset A^a and the weight has been increased accordingly. The *Type I* intervals occur when the

weight has been increased by $2\mathbf{e}_i$ and the *Type II* intervals when the weight has been increased by $\mathbf{e}_i + \mathbf{e}_j$ with $i \neq j \in \mathbb{P}$ (see Figure 2). In the following we denote $\alpha x^i y^j \beta$ a colored permutation where α and β are the starting and trailing colored subwords.

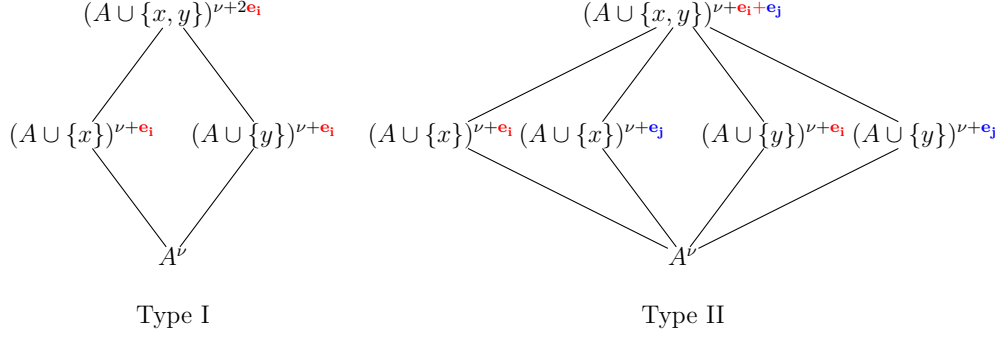


FIGURE 2. Cohomology Relations in \mathbb{B}_n^w

Theorem 3.2. *The set $\{\bar{c}(\sigma) \mid \sigma \in \mathfrak{S}_\mu\}$ is a generating set for $\tilde{H}^{n-2}((\hat{0}, [n]^\mu))$, subject only to the relations for $i \neq j \in \text{supp}(\mu)$*

$$(3.3) \quad \bar{c}(\alpha x^i y^i \beta) + \bar{c}(\alpha y^i x^i \beta) = 0$$

$$(3.4) \quad \bar{c}(\alpha x^i y^j \beta) + \bar{c}(\alpha y^j x^i \beta) + \bar{c}(\alpha y^i x^j \beta) + \bar{c}(\alpha x^j y^i \beta) = 0$$

Proof. By the comments above we know that $\{\bar{c}(\sigma) \mid \sigma \in \mathfrak{S}_\mu\}$ is a set of generators in $\mathcal{M}((\hat{0}, [n]^\mu))$. Observe that the relations (3.3) and (3.4) correspond exactly to the cohomology relations of Type I and Type II respectively and these generate the space of cohomology relations. \square

3.2. The isomorphism. Following relations (1.5) and (1.6) we can conclude a similar proposition for $\Lambda(\mu)$.

Proposition 3.3. *The set $\{\wedge(\sigma) \mid \sigma \in \mathfrak{S}_\mu\}$ is a generating set for $\Lambda(\mu)$ subject only to the relations for $i \neq j \in \text{supp}(\mu)$*

$$(3.5) \quad \wedge(\alpha x^i y^i \beta) + \wedge(\alpha y^i x^i \beta) = 0$$

$$(3.6) \quad \wedge(\alpha x^i y^j \beta) + \wedge(\alpha y^j x^i \beta) + \wedge(\alpha y^i x^j \beta) + \wedge(\alpha x^j y^i \beta) = 0$$

Theorem 3.4. *For each $\mu \in \text{wcomp}_n$, the map $\varphi : \Lambda(\mu) \rightarrow \tilde{H}^{n-2}((\hat{0}, [n]^\mu))$ determined by*

$$\varphi(\wedge(\sigma)) = \bar{c}(\sigma) \text{ for all } \sigma \in \mathfrak{S}_\mu,$$

is an \mathfrak{S}_n -module isomorphism.

Proof. The generators of the two sets $\Lambda(\mu)$ and $\tilde{H}^{n-2}((\hat{0}, [n]^\mu))$ are indexed by colored permutations in \mathfrak{S}_μ and φ maps generators to generators. By Theorem 3.2 and Proposition 3.3, φ also maps relations to relations and clearly respects the \mathfrak{S}_n action. \square

4. HOMOTOPY TYPE OF MAXIMAL INTERVALS IN \mathbb{B}_n^w

4.1. EL-labeling. Let P be a bounded poset. Recall from Section 3 that an edge labeling is a map $\bar{\lambda} : \mathcal{E}(P) \rightarrow \Lambda$, from the set $\mathcal{E}(P)$ of edges of the Hasse diagram P to some fixed poset Λ . Recall also that to any maximal chain $c = (\hat{0} = x_0 < x_1 < \cdots < x_{\ell-1} < x_\ell = \hat{1})$ in P corresponds a word of labels $\bar{\lambda}(c) = \bar{\lambda}(x_0, x_1)\bar{\lambda}(x_1, x_2) \cdots \bar{\lambda}(x_{\ell-1}, x_\ell)$. We say that c is *increasing* if its word of labels $\bar{\lambda}(c)$ is *strictly* increasing, that is, c is increasing if

$$\bar{\lambda}(x_0, x_1) < \bar{\lambda}(x_1, x_2) < \cdots < \bar{\lambda}(x_{\ell-1}, x_\ell).$$

We say that c is *ascent-free* (or decreasing, or falling) if its word of labels $\bar{\lambda}(c)$ has no ascents, i.e. $\bar{\lambda}(x_i, x_{i+1}) \not< \bar{\lambda}(x_{i+1}, x_{i+2})$, for all $i = 0, \dots, \ell-2$. An *edge-lexicographical labeling* (EL-labeling, for short) of P is an edge labeling such that in each closed interval $[x, y]$ of P , there is a unique increasing maximal chain, and this chain lexicographically precedes all other maximal chains of $[x, y]$. See [3] and [4] for more information about EL-labelings.

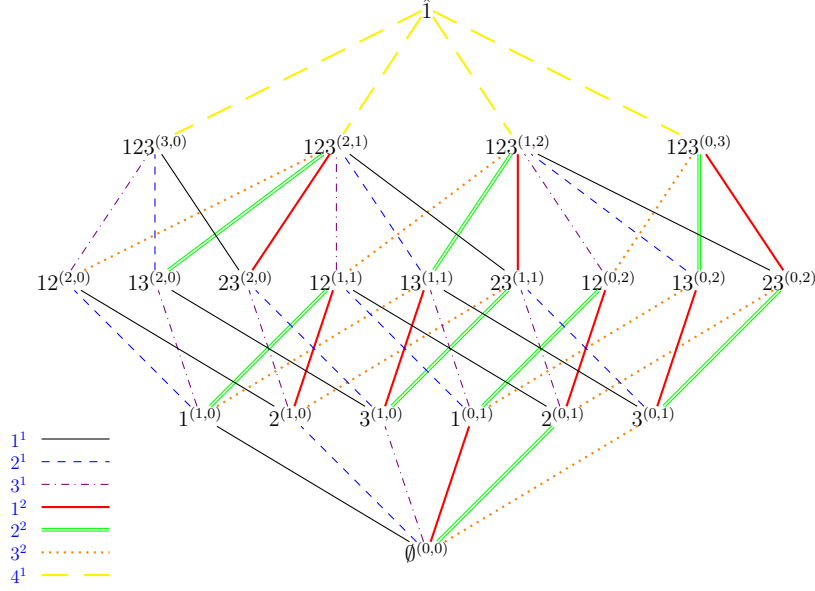
We let $\Lambda = [n+1]^\mathbb{P} := [n+1] \times \mathbb{P}$ be the product poset of the totally ordered sets $[n+1]$ and \mathbb{P} and we define for any $S \subseteq \mathbb{P}$ the labeling $\bar{\lambda} : \mathcal{E}(\widehat{\mathbb{B}_n^S}) \rightarrow [n+1]^\mathbb{P}$ by

$$(4.1) \quad \begin{aligned} \bar{\lambda}(A^\nu, (A \cup \{x\})^{\nu+\mathbf{e}_i}) &= x^i \\ \bar{\lambda}([n]^\mu, \hat{1}) &= (n+1)^1 \end{aligned}$$

In Figure 3 this labeling is illustrated in the case of $\widehat{\mathbb{B}_n^{[2]}}$. The edges have been differentiated by shape and color corresponding to the different labels that appear in the legend. Note that this labeling restricts to the labeling of equation (3.2) in the maximal intervals $[\hat{0}, [n]^\mu]$.

Theorem 4.1. *The labeling $\bar{\lambda} : \mathcal{E}(\widehat{\mathbb{B}_n^S}) \rightarrow [n+1]^\mathbb{P}$ in (4.1) is an EL-labeling of $\widehat{\mathbb{B}_n^S}$.*

Proof. We want to show that in every closed interval of $\widehat{\mathbb{B}_n^S}$ there is a unique increasing chain (from bottom to top), which is also lexicographically first. Note that any interval $[A^\nu, (A \cup B)^{\nu+\mu}]$ in \mathbb{B}_n^S is canonically isomorphic to an interval $[\hat{0}, B^\mu]$ in \mathbb{B}_B^S , where \mathbb{B}_B is the boolean algebra on the set $B \subseteq [n]$, and this isomorphism respects the

FIGURE 3. EL-labeling in $\widehat{\mathbb{B}_3^{[2]}}$

labeling $\bar{\lambda}$. So we only need to care for intervals of the form $[\hat{0}, [n]^\mu]$ and of the form $[\hat{0}, \hat{1}]$.

For Intervals of the form $[\hat{0}, [n]^\mu]$ there is only one possible increasing label word $1^{u_1}2^{u_2}\cdots n^{u_n}$ with $u_1 \leq u_2 \leq \cdots \leq u_n$. This label word is lexicographically first and since, by Proposition 3.1, we know that \mathfrak{S}_μ is in bijection with the set of maximal chains of $[\hat{0}, [n]^\mu]$ only one chain has this label word, that is

$$\hat{0} < \{1\}^{u_1} < \{1, 2\}^{u_1+u_2} < \cdots < [n]^{\sum_{i=1}^n u_i}.$$

For the interval $[\hat{0}, \hat{1}]$ an increasing chain c must be of the form $c' \cup \{\hat{1}\}$, where c' is the unique increasing chain of some interval $[\hat{0}, [n]^\mu]$. All such chains will have last step $n^u(n+1)^1$ and this is only increasing when $u = 1$, so c' is the increasing chain in $[\hat{0}, [n]^{n\mathbf{e}_1}]$. This unique increasing chain has word of labels $1^1 2^1 \cdots (n+1)^1$ that is clearly lexicographically first. \square

The following theorem links lexicographic shellability with topology.

Theorem 4.2 (Björner and Wachs [5]). *Let $\bar{\lambda}$ be an EL-labeling of a bounded poset P . Then for all $x < y$ in P ,*

- (1) the open interval (x, y) is homotopy equivalent to a wedge of spheres, where for each $r \in \mathbb{N}$ the number of spheres of dimension r is the number of ascent-free maximal chains of the closed interval $[x, y]$ of length $r + 2$.
- (2) the set

$$\{\bar{c} \mid c \text{ is an ascent-free maximal chain of } [x, y] \text{ of length } r + 2\}$$
 forms a basis for cohomology $\tilde{H}^r((x, y))$, for all r .

We would like now to characterize the ascent-free chains of the EL-labeling of Theorem 4.1. We already know by Proposition 3.1 that the maximal chains in \mathbb{B}_n^S are in bijection with permutations in \mathfrak{S}_n^S . Since any maximal chain in $\widehat{\mathbb{B}}_n^S$ is of the form $c' \cup \{\hat{1}\}$ where c' is a maximal chain in \mathbb{B}_n^S then the permutations in \mathfrak{S}_n^S are also in bijection with maximal chains in $\widehat{\mathbb{B}}_n^S$. Recall that the set Ninc_n is the set of nonincreasing colored permutations. For $\sigma \in \mathfrak{S}_n^S$ denote $\hat{c}(\sigma) := c(\sigma) \cup \{\hat{1}\}$ and denote by $\overline{\text{Ninc}}_n^S$ the set of permutations in Ninc_n^S with $\text{color}(\sigma(n)) \neq 1$. We have the following characterization of the ascent-free chains.

Theorem 4.3. *The set $\{c(\sigma) \mid \sigma \in \text{Ninc}_\mu\}$ is the set of ascent-free maximal chains of $[\hat{0}, [n]^\mu]$ and the set $\{\hat{c}(\sigma) \mid \sigma \in \overline{\text{Ninc}}_n^S\}$ is the set of ascent-free maximal chains of $\widehat{\mathbb{B}}_n^S$ in the EL-labeling of Theorem 4.1.*

Proof. An ascent in the word of labels of a maximal chain c is of the form

$x^i = \bar{\lambda}(A^\nu, (A \cup \{x\})^{\nu+\mathbf{e}_i}) < \bar{\lambda}((A \cup \{x\})^{\nu+\mathbf{e}_i}, (A \cup \{x, y\})^{\nu+\mathbf{e}_i+\mathbf{e}_j}) = y^j$,
 i.e., $x < y$ and $i \leq j$. This corresponds exactly to the definition of an ascent $x^i y^j$ in a colored permutation. Using Proposition 3.1 we see that ascent-free chains in $[\hat{0}, [n]^\mu$ correspond to colored permutations with no ascents. To describe the ascent-free chains in $\widehat{\mathbb{B}}_n^S$ we now are only missing to check that there is no ascent of the form

$$x^i = \bar{\lambda}([n] \setminus \{x\}^{\mu-\mathbf{e}_i}, [n]^\mu) < \bar{\lambda}([n]^\mu, \hat{1}) = (n+1)^1,$$

which will happen exactly in the case where $i = 1$. \square

We obtain the following corollaries of Theorems 4.1, 4.2, 4.3 and the isomorphism of Theorem 2.1.

Corollary 4.4. *The poset $\widehat{\mathbb{B}}_n^S$ is Cohen-Macaulay and its order complex $\Delta(\mathbb{B}_n^S \setminus \{\hat{0}\})$ has the homotopy type of a wedge of $|\overline{\text{Ninc}}_n^S|$ spheres of dimension $(n-1)$. For every $\mu \in \text{wcomp}$ the interval $[\hat{0}, [n]^\mu]$ is Cohen-Macaulay and its order complex $\Delta([\hat{0}, [n]^\mu])$ has the homotopy type of a wedge of $|\text{Ninc}_\mu|$ spheres of dimension $(n-2)$.*

Corollary 4.5. *The set $\{c(\sigma) \setminus \{\hat{0}\} \mid \sigma \in \overline{\mathbf{Ninc}_n^S}\}$ is a basis for $\tilde{H}^{n-1}(\mathbb{B}_n^S \setminus \{\hat{0}\})$. For every $\mu \in \text{wcomp}$ the set $\{\bar{c}(\sigma) \mid \sigma \in \mathbf{Ninc}_\mu\}$ is a basis for $\tilde{H}^{n-2}((\hat{0}, [n]^\mu))$.*

Corollary 4.6. *For every $\mu \in \text{wcomp}$ the set $\{\wedge(\sigma) \mid \sigma \in \mathbf{Ninc}_\mu\}$ is a basis for $\Lambda(\mu)$. Consequently, $\dim \Lambda(\mu) = |\mathbf{Ninc}_\mu|$.*

Proof of Theorem 2.4. If $\sigma \in \mathfrak{S}_n^{\mathbb{P}}$ is a colored permutation, denote by $\tilde{\sigma} \in \mathfrak{S}_n$ the underlying uncolored permutation associated to σ . For example if $\sigma = 2^1 1^4 3^2$ then $\tilde{\sigma} = 213$. Note that the type $\lambda(\tau)$ defined in Section 1 for a permutation $\tau \in \mathfrak{S}_n$ is closely related to the coloring condition in \mathbf{Ninc}_n . If $\sigma \in \mathbf{Ninc}_n$ such that $\tilde{\sigma} = \tau$ the colors in each part of the partition $\pi(\tau)$ need to strictly decrease from left to right. If B is a block of $\pi(\tau)$ of size $|B| = i$ then the elementary symmetric function $e_i(\mathbf{x})$ enumerates all the possible ways of coloring the letters in B . Then the contribution to the generating function (2.2) of all the nonincreasing colored permutations with underlying uncolored permutation τ is $e_{\lambda(\tau)}(\mathbf{x})$. By Corollary 4.6 and the comments above we have

$$\begin{aligned}
\sum_{\mu \in \text{wcomp}_n} \dim \Lambda(\mu) \mathbf{x}^\mu &= \sum_{\mu \in \text{wcomp}_n} |\mathbf{Ninc}_\mu| \mathbf{x}^\mu \\
&= \sum_{\sigma \in \mathbf{Ninc}_n} \mathbf{x}^{\mu(\sigma)} \\
&= \sum_{\tau \in \mathfrak{S}_n} \sum_{\substack{\sigma \in \mathbf{Ninc}_n \\ \tilde{\sigma} = \tau}} \mathbf{x}^{\mu(\sigma)} \\
&= \sum_{\tau \in \mathfrak{S}_n} e_{\lambda(\tau)}(\mathbf{x}) \quad \square
\end{aligned}$$

5. THE FROBENIUS CHARACTERISTIC OF $\Lambda(\mu)$

To prove Theorem 2.6 we will use a technique introduced by Sundaram [34] (see also [37]) to compute group representations on the (co)homology of Cohen-Macaulay posets. This technique uses the concept of Whitney (co)homology that was introduced by Baclawski in [1]. For information not presented here about symmetric functions and the representation theory of the symmetric group see [24], [29], [20] and [32, Chapter 7].

5.1. A multiplicative inverse formula. Recall that, for some ring R , the ring Λ_R of symmetric functions is the subring of the power series ring $R[[\mathbf{x}]]$ formed by power series of bounded degree that are invariant under the permutation of the variables. We denote by $\widehat{\Lambda}_R$ the ring of

symmetric power series in $R[[\mathbf{x}]]$, that is the completion of Λ_R with respect to the valuation given by the degree.

It is known that the Grothendieck ring of representations of symmetric groups $Rep_{\mathfrak{S}}$ is isomorphic to the ring of symmetric functions $\Lambda_{\mathbb{Z}}$ under the Frobenius characteristic map $ch : Rep_{\mathfrak{S}} \rightarrow \Lambda_{\mathbb{Z}}$. The product in $Rep_{\mathfrak{S}}$ is called the *induction product* and is defined for an \mathfrak{S}_m -module V and an \mathfrak{S}_n -module W by

$$V \circ W := (V \otimes W) \uparrow_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}},$$

where \uparrow_* denotes induction. We will need the following proposition that is one of the main properties that makes ch a ring isomorphism.

Proposition 5.1 ([24]). *Let V be an \mathfrak{S}_m -module and W an \mathfrak{S}_n -module. Then*

$$ch \left((V \otimes W) \uparrow_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}} \right) = ch V ch W.$$

Whitney cohomology (over the field \mathbf{k}) of a poset P with a minimum element $\hat{0}$ can be defined for each integer r as follows:

$$WH^r(P) := \bigoplus_{x \in P} \tilde{H}^{r-2}((\hat{0}, x); \mathbf{k}).$$

In the case of a pure Cohen-Macaulay poset this formula becomes

$$(5.1) \quad WH^r(P) := \bigoplus_{\substack{x \in P \\ \rho(x)=r}} \tilde{H}^{r-2}((\hat{0}, x); \mathbf{k}).$$

If a group G of automorphisms acts on the poset P , this action induces a representation of G on $WH^r(P)$ for every r . From equation (5.1), when P is pure and Cohen-Macaulay, $WH^r(P)$ breaks into the direct sum of G -modules

$$(5.2) \quad WH^r(P) \cong_G \bigoplus_{\substack{x \in P/\sim \\ \rho(x)=r}} \tilde{H}^{r-2}((\hat{0}, x); \mathbf{k}) \uparrow_{G_x}^G,$$

where P/\sim is a set of orbit representatives and G_x the stabilizer of x . The following result of Sundaram [34] can be used to compute characters of G -representations on the (co)homology of pure G -posets.

Lemma 5.2 ([34] Lemma 1.1). *Let P be a bounded poset of length $\ell \geq 1$ and let G be a group of automorphisms of P . Then the following isomorphism of virtual G -modules holds*

$$\bigoplus_{i=0}^{\ell} (-1)^{\ell-i} WH^i(P) \cong_G 0.$$

We know by Theorem 4.1 that for every $\mu \in \text{wcomp}_n$ the poset $[\hat{0}, [n]^\mu]$ is Cohen-Macaulay. We will apply Lemma 5.2 to $[\hat{0}, [n]^\mu]^*$, that is the dual poset of $[\hat{0}, [n]^\mu]$.

Now we specify a set of orbit representatives for the action of \mathfrak{S}_n on $[\hat{0}, [n]^\mu]^*$. Denote by α_η , the weighted subset $[[\eta]]^\eta$ of $[n]$ where $\eta \in \text{wcomp}$ is such that $|\eta| \leq n$. We claim that

$$\{\alpha_\eta \mid \eta \in \text{wcomp} \text{ and } \eta \leq \mu\}$$

is such a set of orbit representatives. To see this, note that any weighted subset $\beta \in [\hat{0}, [n]^\mu]^*$ can be obtained as $\beta = \sigma\alpha_\eta$ for suitable $\eta \in \text{wcomp}$ such that $\eta \leq \mu$. It is also clear that $\alpha_\eta \neq \sigma\alpha_{\eta'}$ for $\eta \neq \eta'$ and for every $\sigma \in \mathfrak{S}_n$. The weighted subset α_η has the Young subgroup $\mathfrak{S}_{|\eta|} \times \mathfrak{S}_{n-|\eta|}$ as stabilizer.

Applying equation (5.2) to $[\hat{0}, [n]^\mu]^*$ we obtain,

$$(5.3) \quad WH^r([\hat{0}, [n]^\mu]^*) \cong_{\mathfrak{S}_n} \bigoplus_{\substack{\eta \in \text{wcomp} \\ \eta \leq \mu \\ |\eta| = n-r}} \tilde{H}^{r-2}((\alpha_\eta, [n]^\mu)) \uparrow_{\mathfrak{S}_{|\eta|} \times \mathfrak{S}_{n-|\eta|}}^{\mathfrak{S}_n}.$$

Note that when $r = 1$ the open interval $(\alpha_\eta, [n]^\mu)$ is the empty poset, hence $\tilde{H}^{r-3}((\alpha_\eta, [n]^\mu))$ is isomorphic to the trivial representation of $\mathfrak{S}_{n-1} \times \mathfrak{S}_1$. When $r = 0$, we have that $\alpha_\eta = [n]^\mu$ and in this case we use the convention that $\tilde{H}^{r-3}((\alpha_\eta, [n]^\mu))$ is isomorphic to the trivial representation of \mathfrak{S}_n .

We apply Lemma 5.2 together with equation (5.3) to obtain the following result.

Lemma 5.3. *For $n \geq 0$ and $\mu \in \text{wcomp}_n$ we have the following \mathfrak{S}_n -module isomorphism*

$$(5.4) \quad \mathbf{1}_{\mathfrak{S}_n} \delta_{n,0} \cong_{\mathfrak{S}_n} \bigoplus_{\substack{\eta \in \text{wcomp} \\ \eta \leq \mu}} (-1)^{|\eta|} \tilde{H}^{n-|\eta|-2}((\alpha_\eta, [n]^\mu)) \uparrow_{\mathfrak{S}_{|\eta|} \times \mathfrak{S}_{n-|\eta|}}^{\mathfrak{S}_n},$$

where $\mathbf{1}_{\mathfrak{S}_n}$ denotes the trivial representation of \mathfrak{S}_n .

Lemma 5.4. *For all $n \geq 0$ and $\eta, \nu \in \text{wcomp}$ with $|\nu| + |\eta| = n$, the following $\mathfrak{S}_{|\eta|} \times \mathfrak{S}_{|\nu|}$ -module isomorphism holds:*

$$\tilde{H}^{|\nu|-2}((\alpha_\eta, [n]^{\nu+\eta})) \cong_{\mathfrak{S}_{|\eta|} \times \mathfrak{S}_{|\nu|}} \mathbf{1}_{\mathfrak{S}_{|\eta|}} \otimes \tilde{H}^{|\nu|-2}((\hat{0}, [[\nu]]^\nu)).$$

Proof. The poset $[\alpha_\eta, [n]^{\nu+\eta}]$ is canonically isomorphic to a product poset $P \times Q$ where P is the $\mathfrak{S}_{|\eta|}$ -poset with the unique element $[[\eta]]^\eta$ and Q is the interval $[\hat{0}, ([n] \setminus [[\eta]])^\nu]$ that is an $\mathfrak{S}_{[n] \setminus [[\eta]]}$ -poset. Furthermore by identifying $\mathfrak{S}_{[n] \setminus [[\eta]]}$ with $\mathfrak{S}_{|\nu|}$ we have that Q and $[\hat{0}, [[\nu]]^\nu]$

are isomorphic $\mathfrak{S}_{|\nu|}$ -posets. The isomorphism of the $\mathfrak{S}_{|\eta|} \times \mathfrak{S}_{|\nu|}$ -posets $[\alpha_\eta, [n]^{\nu+\eta}]$ and $P \times [\hat{0}, [|\nu|]^\nu]$ induces the corresponding $\mathfrak{S}_{|\eta|} \times \mathfrak{S}_{|\nu|}$ -isomorphism in cohomology. The Lemma follows after an application of a version of Künneth's theorem and after noticing that the action of $\mathfrak{S}_{|\eta|}$ on P is trivial. \square

Theorem 5.5. *We have that*

$$\sum_{n \geq 1} \sum_{\mu \in \text{wcomp}_n} \text{ch } \tilde{H}^{n-2}((\hat{0}, [n]^\mu)) \mathbf{x}^\mu = \left(\sum_{n \geq 1} (-1)^n h_n(\mathbf{x}) h_n(\mathbf{y}) \right)^{-1}.$$

Proof. We use the convention that $\tilde{H}^r((\alpha_\eta, [n]^\mu)) = 0$ for all r whenever $\alpha_\eta \not\leq [n]^\mu$. Applying the Frobenius characteristic map ch (in \mathbf{y} variables) to both sides of equation (5.4), multiplying by \mathbf{x}^μ and summing over all $\mu \in \text{wcomp}_n$ with $\text{supp}(\mu) \subseteq [k]$ yields

$$\begin{aligned} \delta_{n,0} &= \sum_{\substack{\mu \in \text{wcomp}_n \\ \text{supp}(\mu) \subseteq [k]}} \mathbf{x}^\mu \text{ch} \left(\bigoplus_{\substack{\eta \in \text{wcomp} \\ \eta \leq \mu}} (-1)^{|\eta|} \tilde{H}^{n-|\eta|-2}((\alpha_\eta, [n]^\mu)) \uparrow_{\mathfrak{S}_{|\eta|} \times \mathfrak{S}_{n-|\eta|}}^{\mathfrak{S}_n} \right) \\ &= \sum_{\substack{\mu \in \text{wcomp}_n \\ \text{supp}(\mu) \subseteq [k]}} \mathbf{x}^\mu \sum_{\substack{\eta \in \text{wcomp} \\ \eta \leq \mu}} (-1)^{|\eta|} \text{ch} \left(\tilde{H}^{n-|\eta|-2}((\alpha_\eta, [n]^\mu)) \uparrow_{\mathfrak{S}_{|\eta|} \times \mathfrak{S}_{n-|\eta|}}^{\mathfrak{S}_n} \right) \\ &= \sum_{\substack{\eta \in \text{wcomp} \\ |\eta| \leq n \\ \text{supp}(\eta) \subseteq [k]}} (-1)^{|\eta|} \sum_{\substack{\nu \in \text{wcomp}_{n-|\eta|} \\ \text{supp}(\nu) \subseteq [k]}} \mathbf{x}^{\eta+\nu} \text{ch} \left(\tilde{H}^{|\nu|-2}((\alpha_\eta, [n]^{\nu+\eta})) \uparrow_{\mathfrak{S}_{|\eta|} \times \mathfrak{S}_{|\nu|}}^{\mathfrak{S}_n} \right). \end{aligned}$$

Using Lemma 5.4, Proposition 5.1, the fact that $\text{ch } \mathbf{1}_n = h_n(\mathbf{y})$ and summing over all $n \geq 0$ we have

$$\begin{aligned} 1 &= \sum_{\substack{\eta \in \text{wcomp} \\ \text{supp}(\eta) \subseteq [k]}} (-1)^{|\eta|} \sum_{\substack{\nu \in \text{wcomp} \\ \text{supp}(\nu) \subseteq [k]}} \mathbf{x}^{\eta+\nu} \text{ch} \left(\mathbf{1}_{\mathfrak{S}_{|\eta|}} \otimes \tilde{H}^{|\nu|-2}((\hat{0}, [|\nu|]^\nu)) \uparrow_{\mathfrak{S}_{|\eta|} \times \mathfrak{S}_{|\nu|}}^{\mathfrak{S}_{|\eta|+|\nu|}} \right) \\ &= \sum_{\substack{\eta \in \text{wcomp} \\ \text{supp}(\eta) \subseteq [k]}} (-1)^{|\eta|} \sum_{\substack{\nu \in \text{wcomp} \\ \text{supp}(\nu) \subseteq [k]}} \mathbf{x}^{\eta+\nu} \text{ch } \mathbf{1}_{\mathfrak{S}_{|\eta|}} \text{ch } \tilde{H}^{|\nu|-2}((\hat{0}, [|\nu|]^\nu)) \\ &= \sum_{\substack{\eta \in \text{wcomp} \\ \text{supp}(\eta) \subseteq [k]}} (-1)^{|\eta|} h_{\mathfrak{S}_{|\eta|}}(\mathbf{y}) \mathbf{x}^\eta \sum_{\substack{\nu \in \text{wcomp} \\ \text{supp}(\nu) \subseteq [k]}} \text{ch } \tilde{H}^{|\nu|-2}((\hat{0}, [|\nu|]^\nu)) \mathbf{x}^\nu \\ &= \left(\sum_{n \geq 0} (-1)^n h_n(\mathbf{y}) h_n(x_1, \dots, x_k) \right) \left(\sum_{n \geq 0} \sum_{\substack{\nu \in \text{wcomp}_n \\ \text{supp}(\nu) \subseteq [k]}} \text{ch } \tilde{H}^{n-2}((\hat{0}, [n]^\nu)) \mathbf{x}^\nu \right). \end{aligned}$$

Letting k be arbitrarily large the proof is completed. \square

Remark 5.6. Theorems 2.1 and 5.5 yield Theorem 2.6 as a corollary.

Note that the power series in Theorems 5.5 and 2.6 belong to $\widehat{\Lambda}_R$ with $R = \Lambda_{\mathbb{Q}}$. Now consider the map $E_1 : \widehat{\Lambda}_R \rightarrow R[[y]]$ defined by:

$$E_1(p_i(\mathbf{y})) = y\delta_{i,1}$$

for $i \geq 1$ and extended multiplicatively, linearly and taking the corresponding limits to all of $\widehat{\Lambda}_R$. Note that E_1 is an algebra homomorphism, or *specialization*, since E_1 is defined on generators. The effect of E_1 can be understood under the following definition of the Frobenius characteristic map. Let V be a representation of \mathfrak{S}_n and χ^V its character, then

$$\text{ch}(V) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \chi^V(\sigma) p_{\lambda(\sigma)}(\mathbf{y}),$$

where $\lambda(\sigma)$ is the cycle type of the permutation $\sigma \in \mathfrak{S}_n$.

We have that

$$E_1(\text{ch } V) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \chi^V(\sigma) E_1(p_{\lambda(\sigma)}(\mathbf{y})) = \chi_V(id) \frac{y^n}{n!} = \dim V \frac{y^n}{n!}.$$

In particular since $h_n(\mathbf{y}) = \text{ch}(\mathbf{1}_{\mathfrak{S}_n})$, the Frobenius characteristic of the trivial representation of \mathfrak{S}_n , we have that $E_1(h_n(\mathbf{y})) = \frac{y^n}{n!}$.

Proof of Theorem 2.5 . Apply E_1 to Theorem 2.6. \square

5.2. Combinatorial interpretation of the multiplicative inverse.

We will use Theorem 2.6 to give an explicit formula for the representation of \mathfrak{S}_n on $\text{ch } \Lambda(\mu)$. To do this we will use a combinatorial interpretation of the multiplicative inverse of an ordinary generating function in terms of words with allowed and forbidden links discovered in different but equivalent forms by Fröberg [12], Carlitz-Scoville-Vaughan [7] and Gessel [14]. The theory outlined in [14] has a more general scope of application but it is equivalent to the simplified description that we present here.

We call a *word* a finite string of elements of some partially ordered set \mathcal{A} that we call an *alphabet* (and call its elements *letters*). For example if $\mathcal{A} = \mathbb{P}$ then 1222143 and 2147 are both examples of words. In particular any label by itself and the empty word \emptyset are considered words. We denote by \mathcal{A}^* the set of words with letters in \mathcal{A} . For any two $w_1, w_2 \in \mathcal{A}^*$ we define the *product* $w_1 w_2$ to be the word constructed by concatenation. For example if $w_1 = 13112$ and $w_2 = 316$ then $w_1 w_2 = 13112316$. Note that the product is associative and so expressions like

$w_1 w_2 \cdots w_k$ are well defined. Note that \mathcal{A}^* together with concatenation is just the *free monoid* generated by \mathcal{A} and for a given ring R we denote by $R\langle\langle\mathcal{A}\rangle\rangle$ the ring of (noncommutative) power series on \mathcal{A} . To $w \in \mathcal{A}^*$ and $a \in \mathcal{A}$ we denote $m_a(w)$ the number of times the letter a appears in w and $|w| := \sum_{a \in \mathcal{A}} m_a(w)$ the *length* of w . We call a *link* the product of two letters. The set of links \mathcal{A}^2 is then in bijection with $\mathcal{A} \times \mathcal{A}$.

Consider a partition of the set \mathcal{A}^2 of links into two parts that we call the *allowed links* $\mathcal{L}(\mathcal{A})$ and *forbidden links* $\overline{\mathcal{L}(\mathcal{A})}$. Let \mathcal{W} be the set of words in \mathcal{A}^* constructed exclusively by the concatenation of allowed links and let $\overline{\mathcal{W}}$ the ones constructed using only forbidden links.

In particular, we consider the empty word and the letters in \mathcal{A} as if they are both in \mathcal{W} and $\overline{\mathcal{W}}$.

Consider the following generating functions in $R\langle\langle\mathcal{A}\rangle\rangle$

$$F(\mathcal{W}) = \sum_{w \in \mathcal{W}} w,$$

$$\overline{F}(\overline{\mathcal{W}}) = \sum_{w \in \overline{\mathcal{W}}} (-1)^{|w|} w.$$

The reader can consult [14] for a proof of the following theorem.

Theorem 5.7 (c.f.[14]). *In $R\langle\langle\mathcal{A}\rangle\rangle$, we have*

$$F(\mathcal{W})\overline{F}(\overline{\mathcal{W}}) = 1.$$

5.3. Explicit description of the \mathfrak{S}_n representation on $\Lambda(n)$. A composition $\alpha = (\alpha_1, \dots, \alpha_\ell)$ is a finite sequence of elements $\alpha_i \in \mathbb{P}$. We say that α is a *composition of n* if $|\alpha| := \sum_i \alpha_i = n$. We denote comp the set of compositions and comp_n the set of compositions of n . We denote $\lambda(\alpha)$ the integer partition obtained from α by reordering its parts in weakly decreasing order.

For (perhaps empty) integer partitions ν and λ such that $\nu \subseteq \lambda$ (that is $\nu(i) \leq \lambda(i)$ for all i), let $S^{\lambda/\nu}$ denote the Specht module of skew shape λ/ν and $s_{\lambda/\nu}$ the Schur function of shape λ/ν . Recall that $s_{\lambda/\nu}$ is the image in the ring of symmetric functions of the specht module $S^{\lambda/\nu}$ under the Frobenius characteristic map ch , i.e., $\text{ch } S^{\lambda/\nu} = s_{\lambda/\nu}$.

A *skew hook* is a connected skew shape that avoids the shape $(2, 2)$. Every skew hook can be described by a comoposition α whose parts are the lengths of the horizontal steps from left to right. We denote by $H(\alpha)$ the skew hook determined by $\alpha \in \text{comp}$. See Figure 4 for an example of the skew hook associated to the composition $\alpha = (3, 2, 1, 1, 3)$. The Specht modules of skew hook shape are called *Foulkes representations* since they were studied by Foulkes in [10].

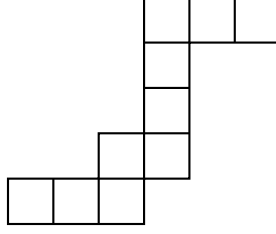


FIGURE 4. Example of the skew hook corresponding to $\alpha = (3, 2, 1, 1, 3)$

Theorem 5.8 (Gessel, personal communication). *We have that*

$$\left(\sum_{n \geq 0} (-1)^n h_n(\mathbf{x}) h_n(\mathbf{y}) \right)^{-1} = \sum_{n \geq 0} \sum_{\alpha \in \text{comp}_n} e_{\lambda(\alpha)}(\mathbf{x}) s_{H(\alpha)}(\mathbf{y}).$$

Proof. Consider the alphabet $\mathcal{A} = \mathbb{P} \times \mathbb{P}$, that is, the set of biletters of the form (a, b) , with partial order given by $(a, b) \leq (c, d)$ if and only if $a \leq c$ and $b \leq d$ (product order). Let

$$\mathcal{L}(\mathcal{A}) = \{(a, b)(c, d) \in \mathcal{A}^2 \mid (a, b) \not\leq (c, d)\}$$

so \mathcal{W} is the set of words of the form

$$(a_1, b_1) \not\leq (a_2, b_2) \not\leq \cdots \not\leq (a_n, b_n),$$

$\overline{\mathcal{W}}$ is the set of words of the form

$$(a_1, b_1) \leq (a_2, b_2) \leq \cdots \leq (a_n, b_n)$$

and we have by Theorem 5.7 that $F(\mathcal{W})\overline{F}(\overline{\mathcal{W}}) = 1$.

Now we consider the \mathbb{Q} -algebra homomorphism (or specialization) $\xi : \mathbb{Q}\langle\langle \mathcal{A} \rangle\rangle \rightarrow \mathbb{Q}[[\mathbf{x}, \mathbf{y}]]$ defined on \mathcal{A}^* by $\xi(w) = \prod_{i=1}^{|w|} x_{w_i(1)} y_{w_i(2)}$ where $w_i = (w_i(1), w_i(2))$ is the i th letter of w . Any word w in $\overline{\mathcal{W}}$ of length $|w| = n$ can be uniquely constructed with a pair of increasing sequences

$$\begin{aligned} a_1 &\leq a_2 \leq \cdots \leq a_n \\ b_1 &\leq b_2 \leq \cdots \leq b_n \end{aligned}$$

and so

$$\xi(\overline{F}(\overline{\mathcal{W}})) = \sum_{n \geq 0} (-1)^n h_n(\mathbf{x}) h_n(\mathbf{y}).$$

Note that $(a_1, b_1) \not\leq (a_2, b_2)$ if and only if either

- (1) $b_1 > b_2$ or
- (2) $b_1 \leq b_2$ and $a_1 > a_2$,

so any word in \mathcal{W} can be uniquely constructed with a pair of sequences:

$$\begin{array}{ccccccc} b_1 \leq \cdots \leq b_{\alpha_1} > & b_{\alpha_1+1} \leq \cdots \leq b_{\alpha_1+\alpha_2} > & \cdots > & b_{\alpha_1+\cdots+\alpha_{\ell-1}+1} \leq \cdots \leq b_{\alpha_1+\cdots+\alpha_\ell} \\ \underbrace{a_1 > \cdots > a_{\alpha_1}}_{\alpha_1} , & \underbrace{a_{\alpha_1+1} > \cdots > a_{\alpha_1+\alpha_2}}_{\alpha_2} , & \cdots , & \underbrace{a_{\alpha_1+\cdots+\alpha_{\ell-1}+1} > \cdots > a_{\alpha_1+\cdots+\alpha_\ell}}_{\alpha_\ell} \end{array}$$

where $\alpha \in \text{comp}$. Then

$$\xi(F(\mathcal{W})) = \sum_{\alpha \in \text{comp}} e_{\lambda(\alpha)}(\mathbf{x}) s_{H(\alpha)}(\mathbf{y}). \quad \square$$

Remark 5.9. Note that essentially the same proof of Theorem 5.8 presented above gives a more general noncommutative version of the identity that reduces to the one stated after applying the specialization that lets the variables \mathbf{x} and \mathbf{y} commute.

If we number the cells of a skew hook H left-to-right and bottom-to-top, i is a *descent* of H if the cell $i+1$ is above the cell i . We denote by $\text{des}(H)$ the descent set of H . For a standard Young tableau of shape λ a *descent* is an entry i that is in a higher row than $i+1$.

Proposition 5.10 (c.f. [38]). *For a skew hook H we have that*

$$s_H(\mathbf{y}) = \sum_{\lambda \vdash n} c_{H,\lambda} s_\lambda(\mathbf{y}),$$

where $c_{H,\lambda}$ is the number of Young tableaux of shape λ and descent set $\text{des}(H)$.

Remark 5.11. Theorems 2.5 and 5.8 together with Proposition 5.10 yield Theorem 2.7 as a corollary.

6. THE KOSZUL PROPERTY

A quadratic associative algebra A and its Koszul dual (co)algebra A^i are said to be *Koszul* if the *Koszul complex* $A^i \otimes_\kappa A$ is *acyclic* (see [23] for the definitions). There are various techniques to conclude the Koszul property of an associative algebra. We use the technique in [26] that involves determining that a family of posets associated to certain types of algebras are Cohen-Macaulay. For a finite subset $S \in \mathbb{P}$ the equivalence classes of colored permutations that generate $\mathcal{S}_S(n)$, considering the symmetry relations (1.3) and (1.4), can be identified with *colored subsets* A^μ , where $A \subseteq [n]$ and $\mu \in \text{wcomp}_{|A|}$ with $\text{supp}(\mu) \in S$. For example $2^1 1^4$, $1^1 2^4$, $2^4 1^1$ and $1^4 2^1$ all represent the same generator in $\mathcal{S}_{[4]}(2)$, and we can represent this generator by $\{1, 2\}^{(1,0,0,1)}$ since the underlying labels are 1 and 2 and exactly one of them has color 1 and

one has color 4. Let $\overline{\mathcal{S}}_S(n)$ be the set of colored subsets whose underlying set is $A = [n]$, the map $[n] \mapsto \overline{\mathcal{S}}_S(n)$ defines a functor (Joyal's species [21]) from the category *Set* of finite sets and bijections to the category \mathfrak{F} of finite sets and arbitrary functions. It is easy to verify that this functor defines a *quadratic* (by relations (1.3) and (1.4)) *cancellative (injective) monoid or c-monoid* in the sense of [26] and that the family of posets associated to $\overline{\mathcal{S}}_S$ in the construction in [26, Section 5.1] is precisely the family of posets \mathbb{B}_n^S for $n \geq 0$. Moreover, the analytic functor $\mathcal{S}_S : \text{Vect}_{\mathbf{k}} \rightarrow \text{Vect}_{\mathbf{k}}$ associated to $\overline{\mathcal{S}}_S$ is $V \mapsto \mathcal{S}_S(V)$.

Theorem 6.1 ([26, Proposition 22, Lemma 40, Theorem 41]). *Let \overline{M} be a c-monoid with associated analytic monoid M . Then M and its Koszul dual M^\dagger are Koszul if and only if the maximal intervals of the poset associated to \overline{M} are Cohen-Macaulay.*

Then the following theorem is a consequence of Theorems 2.2 and 6.1.

Theorem 6.2. *For a finite dimensional vector space V and a finite subset $S \subseteq \mathbb{P}$, the Koszul dual associative algebras $\mathcal{S}_S(V)$ and $\Lambda_S(V)$ are Koszul.*

Remark 6.3. The theory developed in [26] can also be used to conclude the isomorphism of Theorem 2.1. This isomorphism is also a consequence of the fact that the family of posets \mathbb{B}_n^S is the family associated to the c-monoid $\overline{\mathcal{S}}_S$. In Section 3 of this work we went a bit in the longer direction by constructing and providing an explicit isomorphism between the cohomology of the posets associated to this particular c-monoid and its Koszul dual.

Remark 6.4. Note that we obtain as a corollary of Theorem 1.1 the equality of symmetric functions

$$\sum_{\mu \in \text{wcomp}_n} \text{ch } \mathcal{S}(\mu) \mathbf{x}^\mu = h_n(\mathbf{x}) h_n(\mathbf{y}),$$

and then Theorem 2.6 can be written as

$$(6.1) \quad \left(\sum_{n \geq 0} (-1)^n \sum_{\mu \in \text{wcomp}_n} \text{ch } \mathcal{S}(\mu) \mathbf{x}^\mu \right) \left(\sum_{n \geq 0} \sum_{\mu \in \text{wcomp}_n} \text{ch } \Lambda(\mu) \mathbf{x}^\mu \right) = 1.$$

This is not a surprising coincidence since the version of equation (6.1) but with $x_i = 1$ for all i is a natural consequence of the Koszulness of $\mathcal{S}_S(V)$ and $\Lambda_S(V)$ considering only the actions of the symmetric groups \mathfrak{S}_n for all n (see for example [26] or [23]). Equation (6.1) itself is also a consequence of the Koszulness of $\mathcal{S}_S(V)$ and $\Lambda_S(V)$ if we consider

a more general action. For all k , the general linear groups GL_k act on the set of weak compositions of length k and, since each of these actions commutes with the actions of the symmetric groups, then the spaces $\mathcal{S}_{[k]}(n)$ and $\Lambda_{[k]}(n)$ are $GL_k \times \mathfrak{S}_n$ -modules. We then consider the modified characteristic map ch , from the set of polynomial $GL_k \times \mathfrak{S}_n$ representations to the set of symmetric functions in \mathbf{y} with symmetric polynomial coefficients in (x_1, \dots, x_k) , that in an irreducible $GL_k \times \mathfrak{S}_n$ -module of the form $W \otimes V$, with W an irreducible GL_k -module and V an irreducible \mathfrak{S}_n -module, is defined by:

$$\text{ch}(W) := \text{char}_{GL_k}(W) \text{ch}_{\mathfrak{S}_n}(V),$$

where $\text{ch}_{\mathfrak{S}_n}(V)$ is the Frobenius characteristic map considered before in \mathbf{y} variables and $\text{char}_{GL_k}(W)$ is the character map that assigns to a polynomial representation of GL_k a symmetric polynomial in variables x_1, x_2, \dots, x_k .

7. SPECIALIZATIONS

It turns out that Theorems 2.6 and 5.8 are doubly-symmetric generalizations of certain classical identities, including Euler's exponential formula for Eulerian polynomials.

In Section 5 from Theorem 2.6 using specialization E_1 we obtained Theorem 2.5. Together with Theorem 2.4 it can be written in the following form already obtained by the author in [15].

Theorem 7.1 ([15]). *We have*

$$\left(\sum_{n \geq 0} (-1)^n h_n(\mathbf{x}) \frac{y^n}{n!} \right)^{-1} = \sum_{n \geq 0} \left(\sum_{\sigma \in \mathfrak{S}_n} e_{\lambda(\sigma)}(\mathbf{x}) \right) \frac{y^n}{n!},$$

The following Proposition is a standard result in the theory of symmetric functions and can be obtained as a special case of a more general Theorem of Egecioğlu and Remmel in [9].

Proposition 7.2 (c.f. [9, Theorem 2.3]). *For every $n \geq 0$,*

$$h_n(\mathbf{x}) = (-1)^n \sum_{\nu \in \text{comp}_n} (-1)^{\ell(\nu)} e_{\nu}(\mathbf{x})$$

Let $E_2 : \Lambda \rightarrow \mathbb{Q}[t]$ be the map defined by $E_2(e_i(\mathbf{x})) = t$ for all $i \geq 1$ and $E_2(1) = 1$. Since E_2 is defined on the generators e_i it is immediate to check that E_2 is an algebra homomorphism or *specialization*. In the same manner it is also easy to verify that the specialization E_2 extends to a specialization $E_2 : \Lambda[[y]] \rightarrow \mathbb{Q}[t][[y]]$ in the algebra of power series

in y with symmetric function coefficients in Λ (with variables \mathbf{x}) defined by applying E_2 coefficientwise. Note that for any $\lambda \vdash n$ we have that

$$(7.1) \quad E_2(e_\lambda(\mathbf{x})) = t^{\ell(\lambda)}.$$

Lemma 7.3. *For every $n \geq 1$,*

$$(7.2) \quad E_2(h_n(\mathbf{x})) = t(t-1)^{n-1}.$$

Proof. Using Proposition 7.2 and equation (7.1),

$$\begin{aligned} E_2(h_n(\mathbf{x})) &= E_2 \left((-1)^n \sum_{\nu \in \text{comp}_n} (-1)^{\ell(\nu)} e_\nu(\mathbf{x}) \right) \\ &= (-1)^n \sum_{\nu \in \text{comp}_n} (-t)^{\ell(\nu)} \\ &= (-1)^n \sum_{k=1}^n \binom{n-1}{k-1} (-t)^k \\ &= (-1)^n (-t)(1-t)^{n-1}, \end{aligned}$$

where $\binom{n-1}{k-1}$ is the number of compositions of n into k parts. \square

Applying the specialization E_2 to the symmetric function

$$\sum_{\sigma \in \mathfrak{S}_n} e_{\lambda(\sigma)}(\mathbf{x}),$$

using equation (7.1) and the observation that $\ell(\lambda(\theta)) = \text{des}(\theta) + 1$, where $\text{des}(\sigma) = |\{i \in [n-1] \mid \sigma(i) > \sigma(i+1)\}|$, we obtain

$$\begin{aligned} E_2 \left(\sum_{\sigma \in \mathfrak{S}_n} e_{\lambda(\sigma)}(\mathbf{x}) \right) &= \sum_{\sigma \in \mathfrak{S}_n} t^{\ell(\lambda(\sigma))} \\ &= \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)+1} \\ &= A_n(t), \end{aligned}$$

the n -th Eulerian polynomial. The reader can verify now that by applying E_2 to the identity in Theorem 7.1 we obtain the classical Euler's exponential formula.

Theorem 7.4 ([28]). *We have*

$$\frac{1-t}{1-te^{(1-t)y}} = \sum_{n \geq 0} A_n(t) \frac{y^n}{n!}.$$

On the other hand if we apply specialization E_2 directly to Theorem 5.8 we obtain the equation

$$(7.3) \quad \left(1 - h_1(\mathbf{y}) + \sum_{n \geq 2} (-1)^n t(t-1)^{n-1} h_n(\mathbf{y})\right)^{-1} = \sum_{n \geq 0} \sum_{\alpha \in \text{comp}_n} s_{H(\alpha)}(\mathbf{y}) t^{\ell(\alpha)}.$$

The terms in the right-hand side of equation 7.3 have a particular meaning. Indeed, the regular representation $\mathbb{C}[\mathfrak{S}_n]$ of \mathfrak{S}_n decomposes nicely into Foulkes representations.

Theorem 7.5 ([10]). *We have that*

$$h_1(\mathbf{y})^n = \text{ch}(\mathbf{1}_n^{\otimes n} \uparrow_{\mathfrak{S}_1^{\times n}}^{\mathfrak{S}_n}) = \text{ch } \mathbb{C}[\mathfrak{S}_n] = \sum_{\alpha \in \text{comp}_n} s_{H(\alpha)}(\mathbf{y}).$$

If we use Theorem 7.5 and specialize further in equation (7.3) by setting $t = 1$ we obtain

$$(7.4) \quad (1 - h_1(\mathbf{y}))^{-1} = \sum_{n \geq 0} \sum_{\alpha \in \text{comp}_n} s_{H(\alpha)}(\mathbf{y}) = \sum_{n \geq 0} \text{ch}(\mathbb{C}[\mathfrak{S}_n]),$$

indicating that the representation

$$\Lambda_S(n) \simeq_{\mathfrak{S}_n} \bigoplus_{\substack{\mu \in \text{wcomp}_n \\ \text{supp}(\mu) \subseteq S}} \Lambda(\mu)$$

is a generalization of the regular representation $\mathbb{C}[\mathfrak{S}_n]$. In Figure 5 the reader can appreciate the relations between the different identities and specializations discussed in this section.

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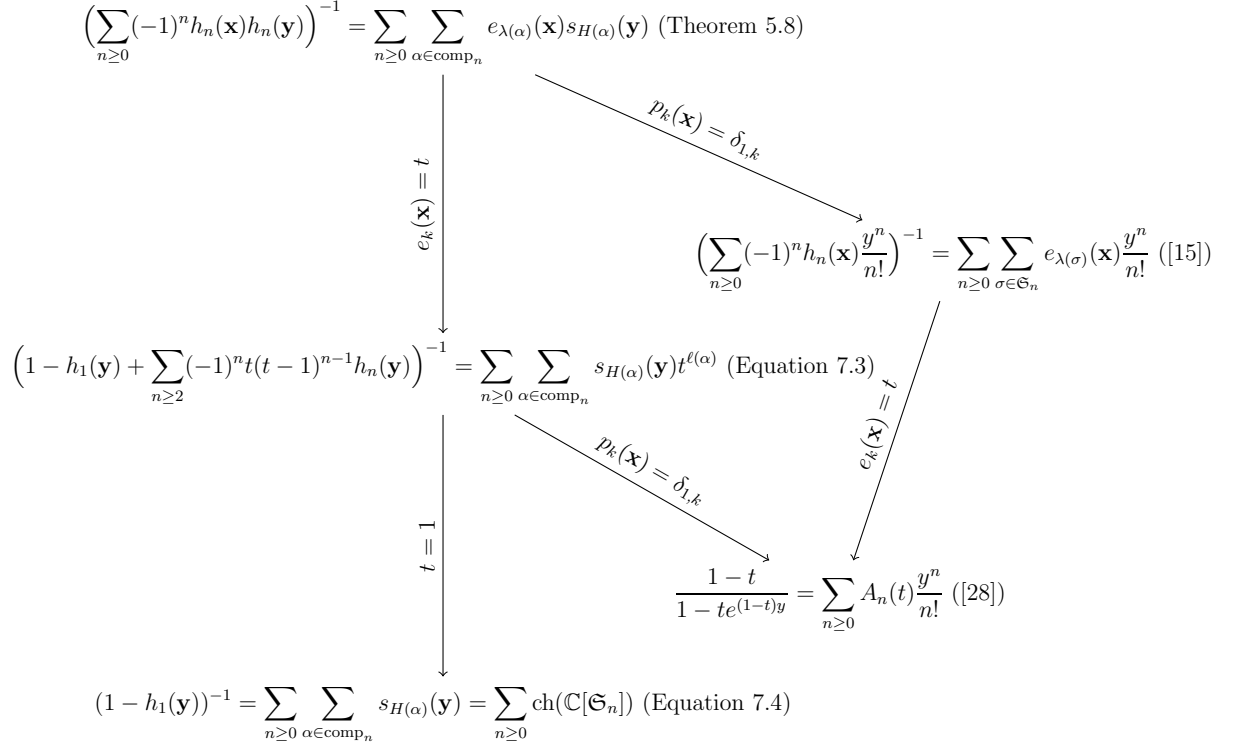


FIGURE 5. Diagram of specializations

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